1. We roll two dices. X is the result of one of them and Z the sum of the results. Find $\mathbf{E}[Z|X]$ and $\mathbf{E}[X|Z]$.

Solution X, Y the results of the roll of the dices. $Z = X+Y$. Y, X are independent

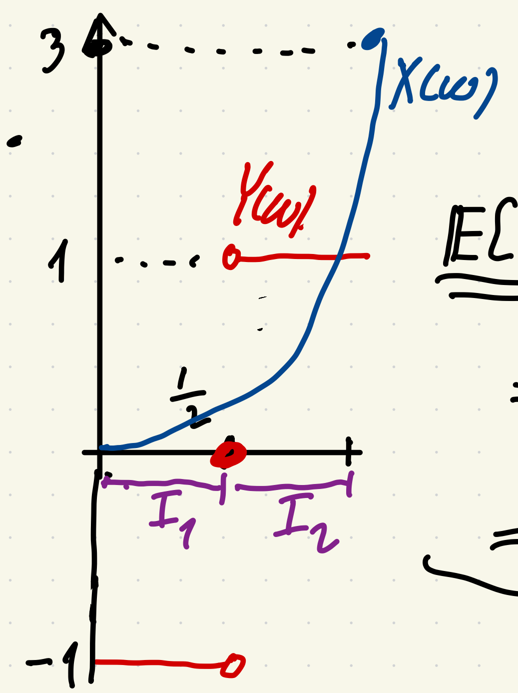
a.) $\mathbb{E}[Z|X] = \underbrace{\mathbb{E}[X|X]}_{X} + \underbrace{\mathbb{E}[Y|X]}_{\mathbb{E}[Y]} = X + \frac{7}{2}$.

b.) $\mathbb{E}[X|Z] = \mathbb{E}[Y|Z]$; $\mathbb{E}[X|Z] + \mathbb{E}[Y|Z] = \mathbb{E}[Z|Z] = Z$

$\mathbb{E}[X|Z] = \frac{Z}{}$

2. **Homework 1.A.** [redacted] Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space where $\Omega = [0,1]$, $\mathcal{F}$ is the Borel σ-algebra and $\mathbf{P}$ is the Lebesgue measure on it. Define the random variables $X(\omega) = 3\omega^2$ and $Y(\omega) = \mathbb{1}(\omega \in [1/2, 1]) - \mathbb{1}(\omega \in [0, 1/2))$ for any $\omega \in \Omega$. What is $\mathbf{E}(X|Y)$?

<u>Solution</u> $Y(\omega) = \begin{cases} -1 & \text{if } \omega \in [0, \frac{1}{2}) \\ 0 & \text{if } \omega = \frac{1}{2} \\ 1 & \text{if } \omega \in (\frac{1}{2}, 1] \end{cases}$.



$\sigma(Y) = \{\emptyset, \Omega, I_1, I_2\}$ (mod 0)

$$\mathbb{E}[X|Y=-1] = \frac{\int_{I_1} 3w^2\,dw}{1/2} = 2\cdot 3\int_0^{\frac{1}{2}} w^2\,dw = 2\cdot[w^3]_0^{\frac{1}{2}} = \frac{2}{8} = \frac{1}{4}$$

$$\mathbb{E}[X|Y=1] = \frac{\int_{\frac{1}{2}}^{1} 3w^2\,dw}{1/2} = 2\cdot[w^3]_{\frac{1}{2}}^{1} = 2\cdot[1 - \tfrac{1}{8}] = \frac{7}{4}$$

$$\mathbb{E}[X|Y] = 1 + \frac{3}{4}Y.$$

3. Let $\Omega = \{-1, 0, +1\}$, $\mathcal{F} = 2^\Omega$ and $\mu(\{-1\}) = \mu(\{0\}) = \mu(\{+1\}) = 1/3$. Consider also the sub-$\sigma$-algebras

$$\mathcal{G} = \{\emptyset, \{-1\}, \{0, +1\}, \Omega\}, \quad \mathcal{H} = \{\emptyset, \{-1, 0\}, \{+1\}, \Omega\}.$$

Let $X : \Omega \to \mathbb{R}$ be the random variable $X(\omega) = \omega$. Compute $\mathbf{E}(\mathbf{E}(X|\mathcal{G})|\mathcal{H})$ and $\mathbf{E}(\mathbf{E}(X|\mathcal{H})|\mathcal{G})$.

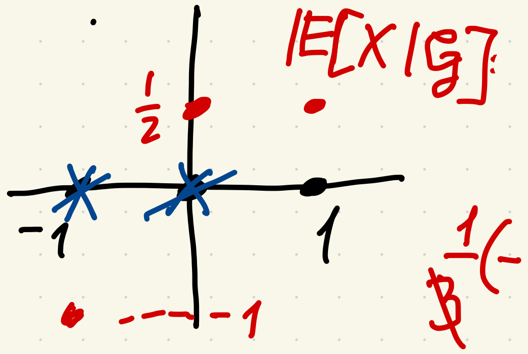
<u>Solution</u> μ is the uniform distribution on $\Omega = \{-1, 0, 1\}$. $\mathcal{G} = \{\phi, \{-1\}, \{0, 1\}, \Omega\}$

$\mathcal{H} = \{0, \{-1, 0\}, \{1\}, \Omega\}$, $X(\omega) := \omega$. $Y := \mathbb{E}[X|\mathcal{G}]$, $Y \in \mathcal{G}$, $\int_G Y d\mu = \int_G X d\mu$ $\quad G \in \mathcal{G}$

$G = \{0, 1\}$; $\quad c \cdot \frac{2}{3} = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = \frac{1}{3} \Rightarrow c = \frac{1}{2}$

$G = \{-1\}$ $\quad d \cdot \frac{1}{3} = -1 \cdot \frac{1}{3} \Rightarrow d = -1$

$\Rightarrow \mathbb{E}[X|\mathcal{G}] = \begin{cases} -1 & \text{on } \{-1\} \\ \frac{1}{2} & \text{on } \{0, 1\} \end{cases}$



$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \begin{cases} -\frac{1}{4} & \text{on } \{-1, 0\} \\ \frac{1}{2} & \text{on } \{1\} \end{cases}$

$\frac{1}{3}(-1) + \frac{1}{3} \cdot \frac{1}{2} = \frac{2}{3} \cdot e \Rightarrow e = -\frac{1}{4}$

The computation of $\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}]$ is similar.

4. Let X_j, $j = 1, 2, \ldots$ be iid. random variables with common distribution $\mathbf{P}(X_j = -1) = \mathbf{P}(X_j = +1) = 1/2$ and let $S_n = X_1 + \cdots + X_n$. Compute the conditional expectations $\mathbf{E}(X_1|S_n)$, $\mathbf{E}(S_n|X_1)$ and $\mathbf{E}(S^2_{n+m}|S_n)$.

Solution $\quad \mathbb{E}[X_i|S_n] = \mathbb{E}[X_j|S_n]$, $i \neq j$, $i, j \leq n$. $\quad \sum_{i=1}^{n} \mathbb{E}[X_i|S_n] = \mathbb{E}[\sum_{i=1}^{n} X_i|S_n] = \mathbb{E}[S_n|S_n] = S_n$

Hence, $\mathbb{E}[X_1|S_n] = \frac{1}{n} S_n$. $\quad \mathbb{E}[S_n|X_1] = \mathbb{E}[X_1|X_1] + \sum_{j=2}^{n} \mathbb{E}[X_j|X_1] = X_1 + (n-1)\underbrace{\mathbb{E}[X_1]}_{0} = X_1$.

$$\mathbb{E}[S^2_{n+m}|S_n] = \mathbb{E}[(S_n + \underbrace{S_{n,m}}_{X_{n+1}+\ldots X_{n+m}})^2|S_n] = \underbrace{\mathbb{E}[S_n^2|S_n]}_{u_1} + 2\underbrace{\mathbb{E}[S_n \cdot S_{n,m}|S_n]}_{u_2} + \underbrace{\mathbb{E}[S^2_{n,m}|S_n]}_{u_3} = S_n^2 + 0 + m - n.$$

$u_1 = S_n^2$; $\quad u_2 = S_n \cdot \mathbb{E}[S_{n,m}|S_n] = S_n \cdot \overbrace{\mathbb{E}[S_{n,m}]}^{0} = 0$.

$$u_3 = \mathbb{E}[S^2_{n,m}|S_n] = \mathbb{E}[S^2_{n,m}] = \mathbb{E}[S_m^2] = \sum_{k=1}^{m} \mathbb{E}[X_k^2] + \sum_{l \neq k} \underbrace{\mathbb{E}[X_l \cdot X_k]}_{0} = \sum_{k=1}^{m} \underbrace{\mathbb{E}[X_k^2]}_{1} = m$$

$$\overbrace{\mathbb{E}[X_l]}^{0} \cdot \overbrace{\mathbb{E}[X_k]}^{0} = 0$$

$$S_{n,m} := X_{n+1} + \ldots + X_{n+m}$$

5. **Homework 1.B.** [redacted] Suppose that the random variables X, Y, Z are jointly defined on a probability space. Prove that

(a) $\mathbf{E}(X) = \mathbf{E}(\mathbf{E}(X|Y))$,

(b) $\mathbf{E}(Y|Z) = \mathbf{E}(\mathbf{E}(Y|X,Z)|Z)$.

Solution We know that $E[X;A]=E[E[X|Y];A]$ if $A\in Y$. We apply this for $A=\Omega$. This proves (a).

Now we prove (b). We use part (a) & $\boxed{E[XY|\mathcal{G}]=Y E[X|\mathcal{G}]}$ (*) for Y bounded and we use $\boxed{E[X]=E[E[X|Y]]}$ (a).

$\mathcal{G}_1=\sigma(Z)$; $\mathcal{G}_2=\sigma(X,Z)$, $\mathcal{G}_1\subset\mathcal{G}_2$. We prove that $\boxed{E[X|\mathcal{G}_1]= E[E[X|\mathcal{G}_2]|\mathcal{G}_1]}$ (**)

TOWER PROPERTY

It is enough to show that

$$\forall A\in\mathcal{G}_1:\quad E\big[E[E[Y|\mathcal{G}_2]|\mathcal{G}_1]\cdot 1\!\!1_A\big]=E\big[E[Y|\mathcal{G}_1]\cdot 1\!\!1_A\big]$$

since conditional expectation is mod 0 unique.

$$E\big[E[E[Y|\mathcal{G}_2]|\mathcal{G}_1]\cdot 1\!\!1_A\big]\overset{(*)\text{ twice}}{=}E\big[E[E[1\!\!1_A\cdot Y|\mathcal{G}_2]|\mathcal{G}_1]\big]\overset{(a)\text{ twice}}{=}E[1\!\!1_A\cdot Y]\overset{(a)}{=}E\big[E[1\!\!1_A\cdot Y|\mathcal{G}_1]\big]$$

$$\overset{(*)}{=}E\big[E[Y|\mathcal{G}_1]\cdot 1\!\!1_A\big].$$

6. Prove the following general version of Bayes's formula. Given the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Let $G \in \mathcal{G}$ and $A \in \mathcal{F}$ with $\mathbf{P}(A) > 0$. Show that

$$\mathbf{P}(G|A) = \frac{\int_G \mathbf{P}(A|\mathcal{G})\, \mathrm{d}\mathbf{P}}{\int_\Omega \mathbf{P}(A|\mathcal{G})\, \mathrm{d}\mathbf{P}}.$$

Solution Let $\begin{matrix} A \in \mathcal{F} \\ \cup \\ G \in \mathcal{G} \end{matrix}$ $\mathbb{P}(A|\mathcal{G}) = \mathbb{E}[\mathbb{1}_A|\mathcal{G}]$ we know by the def. of the conditional expectation that

$$\mathbb{P}(G|A) = \frac{\mathbb{P}(AG)}{\mathbb{P}(A)}$$

$$\mathbb{P}(AG) = \mathbb{E}[\mathbb{1}_A; G] = \mathbb{E}[\overbrace{\mathbb{E}[\mathbb{1}_A|\mathcal{G}]}^{\mathbb{P}(A|\mathcal{G})}, G] = \int_G \mathbb{P}(A|\mathcal{G})\, d\mathbb{P}$$

$$\int_\Omega \mathbb{P}(A|\mathcal{G})\, d\mathbb{P} = \int_\Omega \mathbb{E}[\mathbb{1}_A|\mathcal{G}]\, d\mathbb{P} = \int_\Omega \mathbb{1}_A\, d\mathbb{P} = \mathbb{P}(A)$$

7. Prove the conditional variance formula

$$\mathrm{Var}(X) = \mathbf{E}\left[\mathrm{Var}(X|Y)\right] + \mathrm{Var}\left(\mathbf{E}\left[\mathrm{X}|\mathrm{Y}\right]\right)$$

where $\mathrm{Var}(X|Y) = \mathbf{E}\left[X^2|Y\right] - \left(\mathbf{E}\left[X|Y\right]\right)^2$.

Solution $\mathbb{E}[\mathrm{Var}(X|Y)] = \underbrace{\mathbb{E}[\mathbb{E}[X^2|Y]]}_{\mathbb{E}[X^2]} - \mathbb{E}[(\mathbb{E}[X|Y])^2]$

$+$

$\mathrm{Var}(\mathbb{E}[X|Y]) = \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\underbrace{\mathbb{E}[\mathbb{E}[X|Y]]}_{\mathbb{E}[X]})^2$

$= \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\mathbb{E}[X])^2$

adding up these:

$\mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbb{E}[X|Y]) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathrm{Var}(X)$. ∎

8. Let $X_1, X_2, \ldots$ be iid. random variables and N be a non-negative integer valued random variable which is independent of X_i, $i \geq 1$. Prove that

$$\mathrm{Var}\left(\sum_{i=1}^{N} X_i\right) = \mathbf{E}[N]\,\mathrm{Var}(X) + (\mathbf{E}[X])^2\mathrm{Var}(N).$$

Solution $Z = \sum_{i=1}^{N} X_i$.

$\mathbb{E}[Z \mid N] = ?$

$\mathbb{E}[Z \mid N=n] = \mathbb{E}[\sum_{i=1}^{n} X_i \mid N=n] = n\cdot\mathbb{E}[X]$. So $\mathbb{E}[Z \mid N] = N\cdot\mathbb{E}[X]$.

indep.

$\mathbb{E}[\sum_{i=1}^{n} X_i \mid N=n] = \mathbb{E}[\sum_{i=1}^{n} X_i] = n\,\mathbb{E}[X_1]$

$$\mathrm{Var}[Z \mid N] \overset{\text{def.}}{=} \mathbb{E}[Z^2 \mid N] - (\underbrace{\mathbb{E}[Z \mid N]}_{N\cdot\mathbb{E}[X]})^2.$$

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2 \Big| N=n\right] = \mathbb{E}\left[\sum_{i=1}^{n} X_i^2 + 2\sum_{1\le i<j\le n} X_iX_j \Big| N=n\right] = \mathbb{E}\left[\sum_{i=1}^{n} X_i^2\right] + 2\sum_{1\le i<j\le n}\mathbb{E}[X_i]\,\mathbb{E}[X_j] = n\,\mathbb{E}[X^2] + \cancel{2}\,n(n-1)(\mathbb{E}[X])^2$$

$$\mathrm{Var}[Z \mid N=n] = \mathbb{E}[Z^2 \mid N=n] - (E[Z \mid N=n])^2 = n\cdot\mathbb{E}[X^2] + n(n-1)(\mathbb{E}[X])^2 - n^2(\mathbb{E}[X])^2$$

$$\cancel{n^2(E[X])^2} - n(E[X])^2 - \cancel{n^2(E[X])^2}$$

$$= n(\mathbb{E}[X^2] - (E[X])^2) = n\,\mathrm{Var}(X)$$

$\mathrm{Var}[Z \mid N] = N\cdot\mathrm{Var}(X)$.

$\mathrm{Var}(\mathbb{E}[Z \mid N]) = \mathrm{Var}[N\cdot\mathbb{E}[X]] = (E[X])^2\cdot\mathrm{Var}(N)$. So,

$$\mathrm{Var}(Z) = \mathbb{E}[\mathrm{Var}(Z \mid N)] + \mathrm{Var}(\mathbb{E}[Z \mid N]) = \mathbb{E}[N]\cdot\mathrm{Var}(X) + (E[X])^2\cdot\mathrm{Var}(N).$$

From the previous exercise

9. Let X be a random variable. Assume that Y is another random variable for which $\mathbf{P}(Y=0 \text{ or } Y=1)=1$. Prove that $Y\in\sigma(X)$ iff there exists a $\varphi:\mathbb{R}\to\mathbb{R}$ Borel measurable function such that $Y=\varphi(X)$.

Solution $Y\in\sigma(X) \Longleftrightarrow \exists\, \varphi:\mathbb{R}\to\mathbb{R}$ Borel $Y=\varphi(x)$.

$\Leftarrow$ Let $V\subset\mathbb{R}$ be open. $\varphi^{-1}(V)\in\mathcal{R}$ (Borel). Then $X^{-1}(\varphi^{-1}(V))\in\mathcal{R}$

$\Rightarrow$ $Y\in\sigma(X) \Longrightarrow \{Y=1\}\in\sigma(X)$. Then $\exists$ a Borel set $A\subset\mathbb{R}$ such that $\{Y=1\}=\{X\in A\}$. Then $\mathbb{1}_{\{Y(\omega)=1\}}(\omega)=\mathbb{1}_{\{X(\omega)\in A\}}(\omega)$. Hence $Y(\omega)=\mathbb{1}_A(X(\omega))$.

Let $f:[0,1]\to\mathbb{R}$ be the Cantor function, $g(x):=x+f(x)$. Then $g:[0,1]\to[0,2]$ continuous on $[0,1]$ and $g^{-1}:[0,2]\to[0,1]$ is cont. Let C be the triadic Cantor set. Then $m(g(C))=1$. $\exists$ a non-measurable $D\subset g(C)$. Then $E:=g^{-1}(D)\subset C$. So $m(E)=0$. Let $h(x):=\mathbb{1}_E(x)$. Then

$$h\circ g^{-1}(x)=\begin{cases}1 & \text{if } x\in D\\ 0 & \text{otherwise}\end{cases}$$

So $h\circ g^{-1}$ is NOT Leb. measurable.

h is Leb. measurable, g^{-1} is cont.

On the other hand if $f:\mathbb{R}\to\mathbb{R}$ Leb. measurable & $g:\mathbb{R}\to\mathbb{R}$ is continuous then $g\circ f$ is Leb. measurable.

11. **Homework 1.C.** ([redacted]) Let $Y \in \sigma(\mathcal{G})$. Prove that

$$\mathbf{E}[X|\mathcal{G}] \geq Y \iff \forall A \in \mathcal{G}\ \mathbf{E}[X \cdot \mathbb{1}_A] \geq \mathbf{E}[Y \cdot \mathbb{1}_A].$$

1. $\Rightarrow$ $\mathbb{E}[\mathbb{E}[X|\mathcal{G}];A] = \mathbb{E}[X;A]$. $\mathbb{E}[X;A] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}];A] \overset{\text{assumption}}{\geq} \mathbb{E}[Y;A]$.

$\Leftarrow$ We assume that $\forall A \in \mathcal{G}$ $\boxed{\mathbb{E}[X \cdot \mathbb{1}_A | \mathcal{G}] \geq \mathbb{E}[Y \cdot \mathbb{1}_A | \mathcal{G}]}$. Let $A_\varepsilon := \overbrace{\{\mathbb{E}[X|\mathcal{G}] - Y < -\varepsilon\}}^{\in \mathcal{G}}$.

$0 \leq \mathbb{E}[(X-Y)\cdot \mathbb{1}_{A_\varepsilon}] = \mathbb{E}[X \cdot \mathbb{1}_{A_\varepsilon}] - \mathbb{E}[Y \cdot \mathbb{1}_{A_\varepsilon}] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}] \cdot \mathbb{1}_{A_\varepsilon}] - \mathbb{E}[Y \cdot \mathbb{1}_{A_\varepsilon}] = \mathbb{E}[(\mathbb{E}[X|\mathcal{G}] - Y) \cdot \mathbb{1}_{A_\varepsilon}]$

$\leq \mathbb{P}(A_\varepsilon) \cdot (-\varepsilon)$. This implies that $\mathbb{P}(A_\varepsilon) = 0$, $\forall \varepsilon > 0$.

$\varepsilon_n := \frac{1}{n}$ $\quad A_{\frac{1}{n}} \subseteq A_{\frac{1}{n+1}} \subseteq \dots \Rightarrow A_0 = \bigcup_{n=1}^{\infty} A_{\frac{1}{n}} \Rightarrow \mathbb{P}(A_0) = \lim_{n \to \infty} \mathbb{P}(A_{\frac{1}{n}}) = 0 \Rightarrow \mathbb{P}(\mathbb{E}[X|\mathcal{G}] < Y) = 0.$

1

13. Let $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ satisfying

$$\mathbf{E}[X|Y] = Y \text{ and } \mathbf{E}[Y|X] = X.$$

Show that $\mathbf{P}(X = Y) = 1$.

A simple observation:

If $\mathbb{P}(X \neq Y) > 0$ then either
$\exists c \in \mathbb{Q} : \mathbb{P}(X \leq c, Y > c) > 0$ or
$\exists c \in \mathbb{Q} : \mathbb{P}(Y \leq c, X > c) > 0$.

Solution $\mathbb{E}[X-Y; X>c, Y\leq c] + \mathbb{E}[X-Y; X\leq c, Y\leq c] = \mathbb{E}[X-Y; Y\leq c]$

$\mathbb{P}(X\neq Y) \leq \sum_{c\in\mathbb{Q}} \mathbb{P}(X>c, Y\leq c) + \mathbb{P}(Y>c, X\leq c)$. Fix a $c \in \mathbb{Q}$ we prove:

(*) $\mathbb{P}(X>c, Y\leq c) = \mathbb{P}(Y>c, X\leq c) = 0$

Observe that $\mathbb{E}[X; \{Y\leq c\}] = \mathbb{E}[\underbrace{\mathbb{E}[X|Y]}_{Y}; \underbrace{\{Y\leq c\}}_{\in\sigma(Y)}] = \mathbb{E}[Y; \{Y\leq c\}] \Rightarrow 0 = \mathbb{E}[(X-Y)\cdot \mathbb{1}_{\{Y\leq c\}}]$. By this & this

(**) $0 = \mathbb{E}[(X-Y)\cdot\mathbb{1}_{\{X>c, Y\leq c\}}] + \mathbb{E}[(X-Y)\cdot\mathbb{1}_{\{X\leq c, Y\leq c\}}]$.

Similarly,

$0 = \mathbb{E}[(Y-X)\cdot\mathbb{1}_{\{X\leq c\}}] \Rightarrow$ (***) $0 = \mathbb{E}[(Y-X)\cdot\mathbb{1}_{\{X\leq c, Y>c\}}] + \mathbb{E}[(Y-X)\cdot\mathbb{1}_{\{X\leq c, Y\leq c\}}]$

adding up (**) & (***) the second summands cancel out

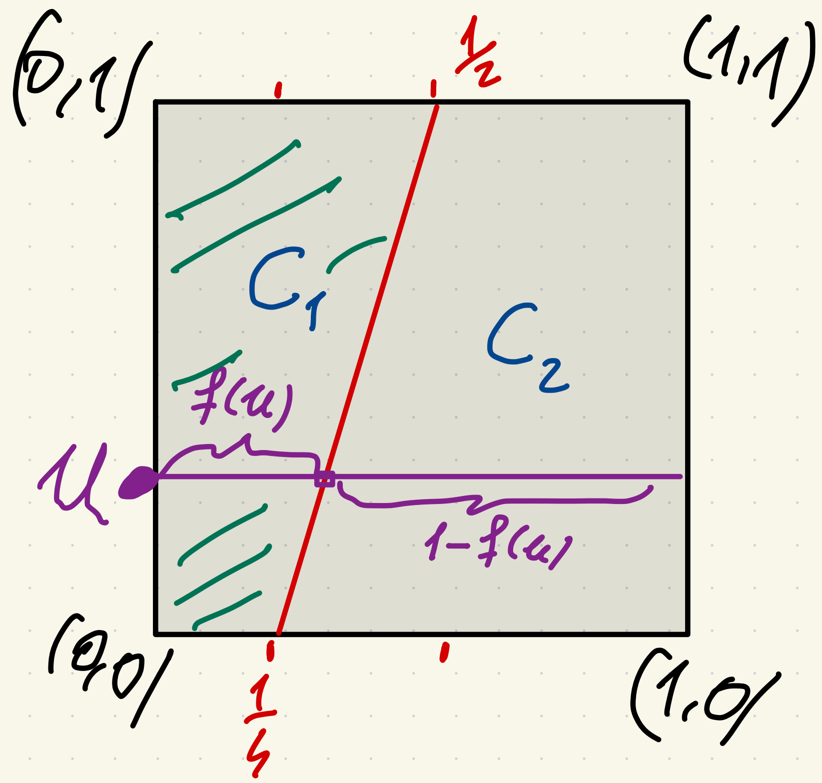
$0 = \overbrace{\mathbb{E}[\underbrace{(X-Y)}_{>0}\cdot\mathbb{1}_{\{X>c, Y\leq c\}}]}^{\geq 0} + \underbrace{\mathbb{E}[\underbrace{(Y-X)}_{>0}\cdot\mathbb{1}_{\{X\leq c, Y>c\}}]}_{\geq 0}$

Hence $\forall c\in\mathbb{Q}$: $0 = \mathbb{P}(X>c, Y\leq c) = \mathbb{P}(Y>c, X\leq c) \Rightarrow \mathbb{P}(X\neq Y) = 0$. ∎

Another example:

$(X, \mathcal{B}, m) = \left([0,1]^2, \mathcal{R}_2, \text{Leb}_2|_{[0,1]^2}\right)$ Borel σ-algebra on $[0,1]^2$

$\mathcal{A} = \{[0,1] \times B : B \in \mathcal{R}([0,1])\}$. Borel σ-alg. of $[0,1]$



Question $\mathbb{E}[\mathbb{1}_{C_1} | \mathcal{A}] = ?$

Solution Let $f(y) = \frac{1}{4}y + \frac{1}{4}$. Then $C_1 = \{(x,y) : 0 \le x \le f(y), y \in [0,1]\}$

Let $G(x,y) := f(y)$. Then G is Borel measurable and constant on every horizontal interval $[0,1] \times \{y\}$. Hence G is $\mathcal{A}$-measurable. Let $U \in \mathcal{A}$. Then $U = [0,1] \times B$, for a $B \in \mathcal{R}_1[0,1]$. Then

$$\iint_U G(x,y)\,dx\,dy = \int_{y\in B} \int_{x=0}^{1} G(x,y)\,dx\,dy = \int_{y \in B} \int_{x=0}^{1} f(y)\,dx\,dy = \int_{y\in B} f(y)\,dy = \text{Leb}_2(U \cap C_1) = \iint_U \mathbb{1}_{C_1}(x,y)\,dx\,dy$$

Hence, $G(x,y)$ is $\mathcal{A}$-measurable $\iint_{[0,1]^2} G(x,y)\,dx\,dy = \iint_{[0,1]^2} \mathbb{1}_{C_1}(x,y)\,dx\,dy$. That is, Leb_2-a.e.

$$\mathbb{E}[\mathbb{1}_{C_1} | \mathcal{A}](x,y) = G(x,y).$$

14. **Homework 2.A.** ([redacted]) Let N_t be the counting process of a Poisson point process with rate $\lambda = 1$. (See section 2.2 in the book R. Durrett: Essentials of Stochastic Processes for the definition.) Find $\mathbf{E}[N_1|N_2]$ and $\mathbf{E}[N_2|N_1]$.

See File C slide 16 in my Stochastic Processes course 2023. This yields that

Solution $N(1) \sim Poi(1)$, $N(2) \sim Poi(2)$, $N(2)-N(1) \sim Poi(1)$, $N(1)$ & $N(2)-N(1)$ are independent.

$N(2) = N(1) + N'(1)$, where $N(1)$ & $N'(1)$ are to i.i.d. $Poi(1)$ r.v. So,

$$\mathbb{E}[N(1)|N(2)] = \mathbb{E}[N(1)|N(1)+N'(1)] = \frac{N(1)+N'(1)}{2} = \frac{N(2)}{2}.$$

$$\mathbb{E}[N(2)|N(1)] = \underbrace{\mathbb{E}[N(2)-N(1)|N(1)]}_{\mathbb{E}[N(2)-N(1)]=1} + \underbrace{\mathbb{E}[N(1)|N(1)]}_{N(1)} = 1+N(1).$$

15. **Homework 2.B.** [redacted] Let $S_n := X_1 + \cdots + X_n$ where $X_1, X_2, \ldots$ are iid. with $X_1 \sim \mathrm{Exp}(1)$. Verify that

$$M_n := \frac{n!}{(1+S_n)^{n+1}} \cdot e^{S_n}$$

is a martingale with respect to the natural filtration $\mathcal{F}_n$.

Solution $0 < M_n$. We want to prove that ⓐ $\boxed{\mathbb{E}[M_n] < \infty, \forall n}$

We know that the sum of n independent $\mathrm{Exp}(\lambda)$ is a $\mathrm{Gamma}(n,\lambda)$ whose density function is $t \mapsto \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$ for $t>0$ and 0 for $t \le 0$. In our case $\lambda = 1$. So, the density function of S_n is $f_n(t) = \begin{cases} e^{-t} \frac{t^{n-1}}{(n-1)!} & \text{if } t>0 \\ 0 & \text{if } t \le 0 \end{cases}$. Hence $\mathbb{E}\left[\frac{n! e^{S_n}}{(1+S_n)^{n+1}}\right] = \int_{t=0}^{\infty} \frac{n! \cancel{e^t}}{(1+t)^{n+1}} \cdot \cancel{e^{-t}} \cdot \frac{t^{n-1}}{(n-1)!} dt$. Hence,

$$\mathbb{E}\left[\frac{n! e^{S_n}}{(1+S_n)^{n+1}}\right] = n \int_0^\infty \left(\frac{t}{1+t}\right)^{n-1} \cdot \frac{1}{(1+t)^2} dt \le n \int_0^\infty \frac{1}{(1+t)^2} dt = n \text{ since } \int_0^\infty (1+t)^{-2} dt = \left[\frac{(1+t)^{-1}}{-1}\right]_0^\infty = 0 - (-1) = 1.$$

So, $\mathbb{E}[M_n] < \infty$. ⓑ M_n is adaptive to $\overset{\mathcal{F}_n}{\sigma(X_1, \ldots, X_n)}$. This is obvious since we can compute the value of M_n from the value of $S_n = X_1 + \cdots + X_n$.

ⓒ $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n, \forall n$. Namely, $\underbrace{}_{X_{n+1} \sim \mathrm{Exp}(1)}$ $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = \mathbb{E}\left[\frac{(n+1)!}{(1+S_n+X_{n+1})^{n+2}} \cdot e^{S_n} \cdot e^{X_{n+1}} \,\middle|\, \mathcal{F}_n\right]$

$$= (n+1)! e^{S_n} \cdot \mathbb{E}\left[\frac{e^{X_{n+1}}}{(1+S_n+X_{n+1})^{n+2}} \,\middle|\, \mathcal{F}_n\right] = (n+1)! e^{S_n} \int_0^\infty \frac{\cancel{e^x}}{(1+S_n+x)^{n+1}} \cdot \cancel{e^{-x}} dx = (n+1)! e^{S_n} \frac{1}{(1+S_n)^{n+1}} \cdot \frac{1}{n+1} = \frac{n! \cdot e^{S_n}}{(1+S_n)^{n+1}} = M_n$$

16. **Homework 2.C.** [redacted]) For every $i = 1, \dots, m$ let $\left\{M_n^{(i)}\right\}_{n=1}^{\infty}$ be a martingale with respect to $\{\mathcal{F}_n\}_{n=1}^{\infty}$. Show that $M_n := \max_{1 \le i \le m} M_n^{(i)}$ is a submartingale with respect to $\{\mathcal{F}_n\}$.

Solution We know that $M_n \in \mathcal{F}_n$ & $\mathbb{E}[|M_n|] < \infty$ by the definition we prove:

(*) $\mathbb{E}[M_{n+1} | \mathcal{F}_n] \geq M_n$.

$M_n^{(i)}$ is a martingale

$$\mathbb{E}[M_{n+1} | \mathcal{F}_n] = \mathbb{E}[\max_{i \le m} M_{n+1}^{(i)} | \mathcal{F}_n] \geq \max_{i \le m} \mathbb{E}[M_{n+1}^{(i)} | \mathcal{F}_n] = \max_{i \le m} M_n^{(i)} = M_n.$$